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Poincaré invariance in effective string theories

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Abstract: We investigate the dispersion relation of the winding closed-string states in $SU(N)$ gauge theory defined on a d-dimensional hypertorus, in a class of effective string theories. We show that order by order in the asymptotic expansion, each energy eigenstate satisfies a relativistic dispersion relation. This is illustrated in the Lüscher-Weisz effective string theory to two-loop order, where the Polyakov loop matrix elements between the vacuum and the closed string states are obtained explicitly. We attempt a generalization of these considerations to the case of compact dimensions transverse to the string.

Keywords: Long strings, Lattice QCD, Confinement.

Contents

1. Introduction

The degrees of freedom and the dynamics of the QCD string have been the object of detailed studies in recent years (see [1] for a review). The energy stored in the gauge field in the presence of a distant static $Q\bar{Q}$ pair grows linearly with their separation R (see e.g. [2, 3] for recent data) in the pure SU(N) theory, suggesting that the flux-lines running between the two color charges behave like a string with a string tension σ.

From this hypothesis, it follows in a natural way that the leading correction to the linear potential corresponds to Gaussian quantum fluctuations of the string around its classical configuration [8], and leads to a regularly spaced low-energy string spectrum with a gap of π/R. This has been verified numerically with remarkable accuracy [4, 9–11].

An elegant way to connect the string-theory predictions with a (gauge-invariant) observable of the gauge theory is to consider the free energy of the $Q\bar{Q}$ pair when inserted inside a thermal heatbath with time extent T. The latter observable is then the Polyakov loop correlator [2, 6] $P^*(\mathbf{x})\ P(\mathbf{y})$, and a systematic expansion in $(\sigma RT)^{-1}$ for its expectation value beyond the area law was proposed in [5]. The free-string partition function reads [4, 5, 14] (in the notation of [5], except for $\tilde{E}_n \to \tilde{M}_n$)

$$\mathcal{Z}_0(T,r) = e^{-\sigma RT-\mu T}\eta(q)^{2-d}, \qquad q \equiv e^{-\pi T/r} \tag{1.1}$$

where

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n). \tag{1.2}$$

Due to the latter function's modular transformation property, $\mathcal{Z}_0$ can be expanded either in terms of eigenstates of the open-string or the closed-string sector [5],

$$\mathcal{Z}_0(T,r) = \sum_{n=0}^{\infty} w_n \, e^{-E_n^0 T} \tag{1.3}$$

$$= e^{-\mu T}\left(\frac{T}{2r}\right)^{\frac{1}{2}(d-2)} \sum_{n=0}^{\infty} w_n \, e^{-\tilde{M}_n^0 r}. \tag{1.4}$$

with

$$E_n^0(r) = \mu + \sigma r + \frac{\pi}{r}\left(-\frac{d-2}{24} + n\right) \tag{1.5}$$

$$\tilde{M}_n^0(T) = \sigma T + \frac{4\pi}{T}\left(-\frac{d-2}{24} + n\right) \tag{1.6}$$

the energies of the open and closed string states respectively. The low-lying closed-string spectrum has also been investigated recently [12, 13]. The next order of the expansion will be discussed below.

The idea of a derivative expansion in the transverse fluctuations was already present in [8]. The work of Polchinski and Strominger [16] was the first to aim at making predictions beyond the universal $1/r$ corrections. It has the merit of being manifestly Poincaré invariant, and to give the possibility of an insight into how the worldsheet degrees of freedom are related to those of the underlying quantum field theory. The more recent approach of Lüscher and Weisz [5] has the nice feature that once the two-dimensional field theory has been postulated and the degrees of freedom identified, the most general interactions, including boundary operators, can be written down automatically. It seems that the Polchinski-Strominger prescription of plugging the induced metric into the Polyakov determinant fixes the coefficient of the $1/r^3$ energy corrections, whereas (in $d = 4$) they are multiplied by an undetermined coefficient in the Lüscher-Weisz effective theory.

In this paper, we address the question whether the proposed effective theory is consistent with Poincaré invariance. Indeed, string theories such as the Nambu-Goto string can be formulated in a form where the latter symmetry is manifest. In an effective string theory the symmetry is in general not manifest. We will however show that to any finite order in the string expansion all closed string states admit a relativistic dispersion relation, $\mathrm{E}^2(p) = M^2 + p^2$ (this property cannot be discussed for the open-string states because they are attached to static charges). The proof is simple and does not rely on the details of the effective theory, but only on the rotation symmetry of infinite space around the worldline of a static quark, and the requirement that open-closed string duality holds order by order in an expansion in powers of 1/distance.

Section 2 describes the setup and gives the proof to the statement made. Section 3 and 4 contain the explicit calculation of the closed-string dispersion relation and Polyakov loop matrix element between the vacuum and the energy eigenstates in the Lüscher-Weisz theory, at leading and at the next order respectively; the constraints on the theory's unknown coefficients are rederived. Section 5 generalizes the discussion to the case of compact

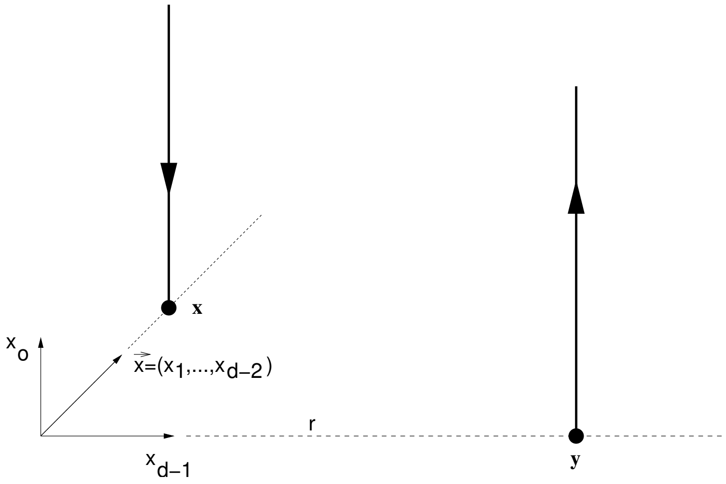

Figure 1: The setup with two Polyakov loops located at $\mathbf{x}$ and $\mathbf{y}$ winding in direction $\mu = 0$.

dimensions transverse to the Polyakov-loop plane. A conclusion summarizes the results obtained.

2. Fourier representation of the Polyakov loop correlator

In this section we focus on the situation where all dimensions, except possibly the one around which the Polyakov loops wind, are much larger than their separation. For two d-dimensional vectors

$$\mathbf{x} \equiv (0, x_1, \ldots, x_{d-2}, 0) \quad \text{and} \quad \mathbf{y} = (0, 0, \ldots, 0, r),$$

we have

$$\langle P^*(\mathbf{x})\ P(\mathbf{y})\rangle = f(\vec{x}, r, T) \tag{2.1}$$

where $\vec{x} = (x_1, \ldots, x_{d-2})$ is a $(d-2)$-dimensional vector and, in the lattice regularized gauge theory,

$$P(x) = \mathrm{Tr}\{U_\mu(x)U_\mu(x + a\hat{\mu}) \ldots U_\mu(x + (T-a)\hat{\mu})\}_{\mu=0}. \tag{2.2}$$

See figure 1. We consider the Fourier representation of f with respect to x:

$$f(\vec{x}, r, T) = \int \frac{d^{d-2}\vec{p}}{(2\pi)^{d-2}}\ c(\vec{p}, r, T)\ e^{i\vec{p}\cdot\vec{x}}, \tag{2.3}$$

The Fourier coefficients c are conversely given by

$$c(\vec{p}, r, T) = \int d^{d-2}\vec{x}\ \ e^{-i\vec{p}\cdot\vec{x}}\ f(\vec{x}, r, T), \tag{2.4}$$

$$= \frac{1}{V_{d-2}}\ \langle\ \mathcal{O}^*(0)\ \mathcal{O}(r)\ \rangle. \tag{2.5}$$

where generically

$$\mathcal{O}(r) \equiv \int d^{d-2}\vec{z}\ \ e^{i\vec{p}\cdot\vec{z}}\ P(0, z_1, z_2, \ldots, z_{d-2}, r) \tag{2.6}$$

and V_{d-2} is the volume of the space parametrized by $x_1, \ldots, x_{d-2}$. These coefficients thus represent correlation functions of Polyakov loops with a definite momentum $\vec{p}$ along directions 1 through $d-2$. On general grounds, the spectral representation of $c(\vec{p}, r, T)$ in the gauge theory (e.g. in terms of eigenstates of the transfer matrix of the lattice-regularized gauge theory along the direction labelled by $d-1$) reads

$$c(\vec{p}, r, T) = \sum_{n\geq 0} |v_n(\vec{p}, T)|^2 \; e^{-\tilde{\mathrm{E}}_n(\vec{p}, T) r}. \tag{2.7}$$

Thus the dispersion relation of the winding flux-tube states can be read off from this quantity.

The $SO(d-1)$ rotational symmetry around one Polyakov loop implies that the correlator is a function of a reduced number of variables:

$$f(\vec{x}, r, T) = \varphi(\sqrt{r^2+\rho^2}, T), \qquad \rho \equiv |\vec{x}|. \tag{2.8}$$

Due to the $SO(d-2)$ symmetry in the dimensions $1, \ldots, d-1$, c obviously depends on $\vec{p}$ only through $|\vec{p}|$, and so do v_n and E_n. How the remaining $SO(d-1)/SO(d-2) \simeq SO(2)$ symmetry is reflected in these functions is the subject of the following section.

2.1 Boost invariance and the two-point function of a local operator

In this section, we drop the T-dependence of c, v_n and E_n, because it will play no role. In particular, what follows applies just as well to the two-point function of QCD operators such as $\mathrm{Tr}\, F_{\mu\nu}^2$, $\bar{\psi}\Gamma\psi$, etc. in $d-1$ dimensions. Starting from eq. (2.4) with f given by (2.8), it is easy to show that

$$\frac{\partial}{\partial r}\frac{\partial}{\partial p}\; c(p, r) = pr\; c(p, r) \tag{2.9}$$

Conversely, if $c(\vec{p}, r)$ satisfies eq. (2.9), it is clear that the function f defined by eq. (2.3) satisfies $(\rho\partial_r - r\partial_\rho) f(\rho, r) = 0$, which implies that f is of the form (2.8)[1]. Thus no information about the symmetry has been lost in going from (2.8) to (2.9).

For a function c admitting an absolutely convergent spectral representation as in (2.7), eq. (2.9) is equivalent to

$$\sum_{n\geq 0} e^{-\tilde{\mathrm{E}}_n(p) r} \left[|v_n(p)|^2 \left(1 - \frac{d}{d(p^2)} \tilde{\mathrm{E}}_n^2(p) \right) + \frac{2}{r}\frac{d}{d(p^2)} \left(|v_n(p)|^2 \tilde{\mathrm{E}}_n(p) \right) \right] = 0, \quad \forall (p, r). \tag{2.10}$$

In particular, dividing by $e^{-\tilde{\mathrm{E}}_0(p) r}$, and letting $r \to \infty$, only the $n = 0$ term remains. In this way we learn that, for $n = 0$,

$$\frac{d}{d(p^2)} \tilde{\mathrm{E}}_n^2(p) = 1, \quad \text{i.e.} \quad \tilde{\mathrm{E}}_n^2(p) = \tilde{\mathrm{E}}_n^2(0) + p^2, \tag{2.11}$$

and the momentum dependence of the matrix elements v_0 is also fixed:

$$|v_n(p)|^2 = \frac{b_n}{\tilde{\mathrm{E}}_n(p)}. \tag{2.12}$$

[1] Indeed, f is then invariant under an infinitesimal rotation $(\rho, r) \to (\rho + \theta r, r - \theta\rho)$.

Thus the $n = 0$ term does not contribute to eq. (2.10). One may then iterate the argument for $n = 1, 2, \ldots$, showing that eq. (2.11) and (2.12) hold for all states. If say a double degeneracy occurs, one can only conclude that the sum of the $|v_n(p)|^2$ are inversely proportional to the common energy. However exact degeneracies normally only occur in quantum field theory for symmetry reasons, in which case one also expects the $|v_n(p)|^2$ of the two states to be equal.

One may draw two conclusions: the states that contribute to a correlator such as c defined in eq. (2.4) with $f(\vec{x}, r)$ of the form (2.8) necessarily admit a relativistic dispersion relation. Conversely, for relativistic $|v_n(p)|^2$ and $\tilde{\mathrm{E}}_n(p)$, $f(\vec{x}, r)$ is of the form (2.8); which implies in particular a central potential for the $Q\bar{Q}$ pair.

It is instructive to ask oneself which step of the argument fails to hold in the case of a non-relativistic theory. In that case, the Euclidean time-direction plays a special role and cannot be interchanged for a spatial coordinate. Thus if x_0 is taken to be the time-direction, then eq. (2.8) holds by spatial rotation symmetry, but eq. (2.7) does not hold in general, and even if it does the $\tilde{E}_n(p)$ are not the eigenvalues of the Hamiltonian, which governs the evolution of states in the x_0 direction rather than in the x_{d-1} direction. Alternatively, if it is x_{d-1} that plays the role of time, then eq. (2.8) does not hold (ρ is now a spatial and r a temporal separation), precisely because the theory does not have the relativistic boost invariance.

2.2 Boost invariance in an effective string theory

As we shall see in the explicit calculations of sections 3 and 4, the Lüscher-Weisz effective string theory, worked out to order m, generically produces an expression for $c(p, r)$ of the form

$$c_m(p, r) = \sum_n e^{-\tilde{\mathrm{E}}_n(p) r} \sum_{k=-k_m}^{\infty} \frac{\gamma_{nk}^{(m)}(p)}{r^k}. \tag{2.13}$$

Here we assume that this correlator has been obtained by performing an integral of the type (2.4) with $f(\vec{x}, r)$ of the form (2.8). Then eq. (2.9) implies $\beta_{nk}(p) = 0\ \forall n, k$ where

$$\begin{aligned}\beta_{nk}(p) \equiv \gamma_{nk}^{(m)}(p)\left(1 - \frac{d}{d(p^2)}\tilde{\mathrm{E}}_n^2(p)\right) + \frac{2}{\tilde{\mathrm{E}}_n(p)^{1-k}}\frac{d}{d(p^2)}\left(\tilde{\mathrm{E}}_n(p)^{2-k}\gamma_{n,k-1}^{(m)}(p)\right)\\ +2(k-2)\frac{d}{d(p^2)}\gamma_{nk-2}^{(m)}(p), \qquad k \geq -k_m\end{aligned} \tag{2.14}$$

with the understanding $\gamma_{n,-k_m-1}^{(m)} = \gamma_{n,-k_m-2}^{(m)} = 0$. For $k = -k_m$, the equation teaches us that $\tilde{\mathrm{E}}_n(p)^2 = \tilde{M}_n^2 + p^2$.

It is a straightforward exercise to show by induction that these conditions imply

$$\gamma_{nk}^{(m)}(p) = \tilde{\mathrm{E}}_n(p)^{k-1} \sum_{j=0}^{k+k_m} \frac{\alpha_{n,k-j}^{(m)}(k-1)(k-2)\ldots(k-2j)}{j!(2\tilde{\mathrm{E}}_n^2(p))^j} \tag{2.15}$$

where the $\alpha_{nk}^{(m)}$ $(k \geq -k_m)$ are real numbers.

2.2.1 Energy corrections

The computational scheme is meant to hold order by order in powers of inverse distance, so that the terms with positive powers of r in $c(p,r)$ must be interpreted as giving the energy shifts of the closed-string states. It is not *a priori* obvious that the argument of section 2.1 goes through, since the terms of the exponential that are missing are not uniformly small in r. We will nevertheless show that the energy shifts preserve the relativistic dispersion relation.

If the scheme is to be at all consistent, the energy shifts must be given by the ratio of the r^1 to the r^0 coefficients:

$$\delta\tilde{\mathrm{E}}_n^{(m)}(p) = -\frac{\gamma_{n,-1}^{(m)}(p)}{\gamma_{n,0}^{(m)}(p)} \tag{2.16}$$

The understanding is that the fraction should be expanded in powers of $\tilde{\mathrm{E}}_n(p)^{-1}$ and that the terms beyond the k_m^{th} power ought to be neglected. For the quadratic, cubic, etc. terms to be consistent with the Taylor expansion of the exponential function, one finds that the numerical coefficients $\alpha_{nk}^{(m)}$ must be related by

$$\frac{\alpha_{n,-k}^{(m)}}{\alpha_{n0}^{(m)}} = \frac{1}{k!}\left(\frac{\alpha_{n,-1}^{(m)}}{\alpha_{n0}^{(m)}}\right)^k, \qquad k \geq 0. \tag{2.17}$$

Thus

$$\gamma_{n,-k}^{(m)}(p) = \frac{\alpha_{n0}^{(m)}}{\tilde{\mathrm{E}}_n(p)^{k+1}k!}\left(\frac{\alpha_{n,-1}^{(m)}}{\alpha_{n0}^{(m)}}\right)^k \sum_{j=0}^{k_m-k}\frac{(2j+k)!}{j!(j+k)!}\left(\frac{\alpha_{n,-1}^{(m)}}{2\alpha_{n0}^{(m)}\tilde{\mathrm{E}}_n^2(p)}\right)^j \qquad 0 \leq k \leq k_m. \tag{2.18}$$

On the other hand, if $E(p)$ satisfies the relativistic dispersion relation, then so does $E(p)(1+x(p))$ if and only if

$$x(p) = -1 + \sqrt{1 + x_0(x_0+2)(M/E(p))^2}, \tag{2.19}$$

where $x_0 \equiv x(0)$ and $M = E(p=0)$. It is now a lengthy but straightforward exercise to check that eq. (2.16) and (2.18) satisfy this condition up to order $1/\tilde{\mathrm{E}}_n^2(p)^{k_m+1}$ corrections and as a by-product one finds

$$-\frac{\delta\tilde{M}_n^{(m)}}{\tilde{M}_n} = \sum_{j=1}^{k_m}\left(\frac{\alpha_{n,-1}^{(m)}}{\alpha_{n0}^{(m)}\tilde{M}_n^2}\right)^j \frac{(2j-3)!!}{j!}. \tag{2.20}$$

(with the convention $(-1)!! = 1$). At this stage, one may consider that the series is actually of the form (2.21), where the energy levels must now be understood as the ones including the corrections (2.20). The analysis of the next section thus applies.

In summary, the type of expressions one obtains for $c(p,r)$ from a long-distance effective theory for a sector of very massive states is such that the latter automatically admit a relativistic dispersion relation as soon as the expression is made consistent with the form expected from a spectral representation. In general these consistency requirements will constrain the parameters of the effective theory. We saw that rotation invariance implies that the $\gamma^{(m)}_{n,-k}$, $0 \le k \le k_m$ are parametrized in terms of (k_m+1) numbers $\alpha^{(m)}_{n,-k}$, $0 \le k \le k_m$. The requirement of the exponentiation of these terms leads to $k_m - 1$ conditions, so that to any order, there are exactly two free parameters per energy level, $\alpha^{(m)}_{n0}$ and $\alpha^{(m)}_{n,-1}$, which determine the overlap and energy of each state n. If the effective theory can satisfy these $k_m - 1$ conditions, the scheme is thus consistent to all orders.

2.2.2 Cancelling powers of $1/r$

We now suppose that the string-theory parameters have been tuned so as to give a definite-momentum correlator of the form

$$c_m(p,r) = \sum_n e^{-\tilde{\mathrm{E}}^{(m)}_n(p)r} \sum_{k=0}^{\infty} \frac{\gamma^{(m)}_{nk}(p)}{r^k}. \tag{2.21}$$

with the $\tilde{\mathrm{E}}^{(m)}_n(p)$ satisfying the relativistic dispersion relation. The functional form is only consistent with the spectral representation (2.7) if one is able to further tune the free parameters of the theory to cancel the terms $k = 1, \ldots, k_m$, and one then neglects the terms with $k > k_m$. This tuning is possible for the first two orders of the Lüscher-Weisz theory (see [5] and the next two sections). Eq. (2.9) applied to the form (2.21) leads to eq. (2.14) with $k_m = 0$. As we assume $\gamma^{(m)}_{n0}(p) \neq 0$, $\beta_{n0}(p) = 0$ requires the relativistic dispersion relation (2.11), and the next equation then implies (2.12). At $k = 1$, one finds that $\gamma^{(m)}_{n1}(p)$ is independent of p; note that the boost symmetry does not imply a relation between the $\alpha_{n,k>0}$ and $\alpha_{n,k\le 0}$. Further on, it is clear that if $\gamma^{(m)}_{nk}(p) = 0$ for $k = 1, \ldots, m-1$, then the momentum dependence of the next coefficient is very simple:

$$\gamma^{(m)}_{nk}(p) = \gamma^{(m)}_{nk}(0) \left(\frac{\tilde{\mathrm{E}}^{(m)}_n(p)}{\tilde{M}_n} \right)^{k-1}. \tag{2.22}$$

In particular, it vanishes identically if and only if it vanishes at $p = 0$.

The conclusion is that for any effective theory of $f(\vec{x}, r, T)$ which produces an expression of the type (2.21) with $\gamma^{(m)}_{n0}(p) \neq 0$, rotation symmetry automatically implies the relations $\tilde{\mathrm{E}}^m_n(p) = \sqrt{p^2 + \tilde{(}M^m_n)^2}$ and $\gamma^{(m)}_{n0}(p) \propto (\tilde{\mathrm{E}}^m_n(p))^{-1}$ and also determines the momentum dependence of $\gamma^{(m)}_{nk\ge 1}(p)$. Cancelling systematically the latter in the effective theory thus allows one to interpret $\tilde{\mathrm{E}}_n(p)$ as the energies of relativistic closed strings. This can be done order by order consistently with Euclidean rotation symmetry, the technical reason being that the derivative w.r.t. and the multiplication by r in the identity (2.9) act locally on the power series in $1/r$.

3. The free bosonic string theory

The effective theory (1.1) makes a prediction for the Polyakov loop correlator (2.1) for any separation. In this section we exploit this fact to compute the Polyakov correlators of definite momentum explicitly, thus extracting the predicted dispersion relation for the closed string states.

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4 dimensions

Since the spatial dimensions have $SO(3)$ as symmetry group, $c_0(\vec{p}, r, T)$, the free-string theory prediction for $c(\vec{p}, r, T)$, can be computed straightforwardly:

$$c_0(\vec{p}, r, T) = e^{-\mu T} \int d^2\vec{x}\ e^{-i\vec{p}\cdot\vec{x}} \frac{T}{2\sqrt{|\vec{x}|^2 + r^2}} \sum_{n\geq 0} w_n\ e^{-\tilde{M}_n^0\sqrt{r^2+|\vec{x}|^2}}. \tag{3.1}$$

We obtain

$$c_0(p, T, r) = e^{-\mu T}\ \pi T \sum_{n\geq 0} w_n\ \frac{1}{\sqrt{p^2 + (\tilde{M}_n^0)^2}} e^{-r\sqrt{p^2+(\tilde{M}_n^0)^2}}, \tag{3.2}$$

from which we read off

$$d = 4: \qquad \tilde{\mathrm{E}}_n^0(p, T) = \sqrt{p^2 + (\tilde{M}_n^0)^2} \tag{3.3}$$

$$|v_n(p, T)|^2 = e^{-\mu T}\ \frac{\pi T w_n}{\sqrt{p^2 + (\tilde{M}_n^0)^2}}. \tag{3.4}$$

As expected, the effective string theory predicts exactly a free, relativistic dispersion relation for the closed string states. The situation is special in four dimensions in that there are no inverse powers of r appear in $c(p, r, T)$ to this order.

3 dimensions

Repeating the exercise for $d = 3$,

$$c_0(p, r, T) = e^{-\mu T} \int_{-\infty}^{\infty} dx\ e^{-ipx} \left(\frac{T}{2\sqrt{r^2 + x^2}}\right)^{1/2} \sum_{n\geq 0} w_n e^{-\tilde{M}_n^0\sqrt{r^2+x^2}}, \tag{3.5}$$

we find

$$c_0(p, r, T) = e^{-\mu T}\ r\ \sqrt{\frac{T}{4\pi}} \sum_{n\geq 0} w_n\ \frac{(\tilde{M}_n^0)^{3/2}}{\sqrt{p^2 + (\tilde{M}_n^0)^2}} \left(K_{\frac{3}{4}}(z_n^+)K_{\frac{1}{4}}(z_n^-) + K_{\frac{3}{4}}(z_n^-)K_{\frac{1}{4}}(z_n^+)\right) \tag{3.6}$$

where

$$z_n^\pm \equiv \frac{r}{2}\left(\sqrt{(\tilde{M}_n^0)^2 + p^2} \pm |p|\right). \tag{3.7}$$

At this stage one can verify that expression (3.6) satisfies (2.9). Using the asymptotic expansion for the modified Bessel functions yields

$$c_0(p, r, T) \sim e^{-\mu T}\ \sqrt{\pi T} \sum_{n\geq 0}\ w_n\ \sqrt{\frac{\tilde{M}_n^0}{(\tilde{M}_n^0)^2 + p^2}}\ e^{-r\sqrt{(\tilde{M}_n^0)^2+p^2}} \left(1 + \frac{\sqrt{(\tilde{M}_n^0)^2 + p^2}}{8r(\tilde{M}_n^0)^2} + \cdots\right). \tag{3.8}$$

Keeping only the leading order term, we have:

$$d = 3: \qquad \tilde{\mathrm{E}}_n^0(p, T) = \sqrt{p^2 + (\tilde{M}_n^0)^2} \tag{3.9}$$

$$|v_n(p, T)|^2 = e^{-\mu T}\ \sqrt{\frac{\pi T \tilde{M}_n^0}{p^2 + (\tilde{M}_n^0)^2}}. \tag{3.10}$$

Thus the leading asymptotic form of c_0 still satisfies eq. (2.10).

4. The Lüscher-Weisz theory at two-loop order

In this section we shall compute the leading non-trivial corrections to the spectrum and matrix elements in the closed string sector predicted by the Lüscher-Weisz effective theory. Ref. [5] gives the general expression for the partition function at two-loop order:

$$\mathcal{Z}(r,T) = \sum_{n\geq 0} w_n \{1 - [E_n^1]T\} e^{-E_n^0 T}, \tag{4.1}$$

with $[E_n^1]$ the weighted average of the first-order energy shifts $E_{n,i}^1$ within the n^{th} free-string multiplet.

4 dimensions

For general values of the 'low-energy constants' (called c_2 and c_3), the corresponding closed-string expansion reads [5]

$$\mathcal{Z}(r,T) = e^{-\mu T}\frac{T}{2r}\sum_{n\geq 0} w_n\; e^{-\tilde{M}_n^0 r}\left\{1 - [\tilde{M}_n^1]r + (2c_2+c_3)\left[\frac{4\pi}{T^2}(n-\frac{1}{12}) + \frac{1}{2Tr}\right]\right\}. \tag{4.2}$$

Starting from this expression, we can compute as before the expression of $c_1(p,r,T)$, the Fourier coefficient of $\mathcal{Z}(r,T)$ with respect to the transverse coordinates:

$$c_1(p,r,T) = e^{-\mu T}\;\pi r T\;\sum_{n\geq 0} w_n\;\times \tag{4.3}$$

$$\left\{1 + r[\tilde{M}_n^1]\frac{\partial}{\partial u} + (2c_2+c_3)\frac{4\pi}{T^2}(n-\frac{1}{12}) + \frac{2c_2+c_3}{2Tr}\int_{r\tilde{M}_n^0}^{\infty} du\right\}_{u=r\tilde{M}_n^0}^{v=pr} I^{(4)}(u,v)$$

with

$$I^{(4)}(u,v) = \frac{e^{-\sqrt{u^2+v^2}}}{\sqrt{u^2+v^2}}. \tag{4.4}$$

What powers of r appear in this expression? To answer this question one may first inspect the $v=0$ case (zero momentum). It is easy to see that, while the free theory term gives a contribution $O(r^0)$, the second term makes contributions $O(r^0)$ and $O(r^{+1})$, and the fourth one produces a whole series of inverse powers of r starting at r^{-1}. While positive powers of r, once exponentiated, give us the energy correction, the $1/r$ term is inconsistent with the spectral representation of c. Therefore, requiring that such terms be absent yields the constraint $2c_2 + c_3 = 0$ (already obtained in [5]). It is remarkable that there are then no inverse powers of r left at all.

For this special set of parameters, the closed-string energy corrections are [5]

$$d=4: \qquad \tilde{M}_n^1(T) = \tilde{M}_n^0 + [\tilde{M}_n^1], \qquad [\tilde{M}_n^1](T) = -c_2\frac{(4\pi)^2}{T^3}(n-\frac{1}{12})^2. \tag{4.5}$$

The final result is

$$c_1(p,r,T) = \sum_{n\geq 0} |v_n^1(p,T)|^2\; e^{-r\tilde{\mathrm{E}}_n^1(p,T)} \;+\; O([\tilde{M}_n^1]^2) \tag{4.6}$$

where

$$\tilde{\mathrm{E}}_n^1(p,T) \;=\; \sqrt{p^2+(\tilde{M}_n^0)^2} + \frac{\tilde{M}_n^0\;[\tilde{M}_n^1]}{\sqrt{p^2+(\tilde{M}_n^0)^2}} \tag{4.7}$$

$$|v_n^1(p,T)|^2 \;=\; |v_n^0(p,T)|^2\;\left(1-\frac{\tilde{M}_n^0\;[\tilde{M}_n^1]}{p^2+(\tilde{M}_n^0)^2}\right). \tag{4.8}$$

Eq. (4.7) implies that the relativistic dispersion relation is still satisfied to this order:

$$(\tilde{\mathrm{E}}_n^1(p,T))^2 = p^2+(\tilde{M}_n^1)^2 \;+\; O([\tilde{M}_n^1]^2), \quad \forall n. \tag{4.9}$$

Furthermore, we have

$$|v_n^1(p,T)|^2\;\tilde{\mathrm{E}}_n^1(p,T) = |v_n^0(p,T)|^2\;\tilde{\mathrm{E}}_n^0(p,T)\;\left[1+O\left(\left(\frac{[\tilde{M}_n^1]}{\tilde{M}_n^0}\right)^2\right)\right], \quad \forall n. \tag{4.10}$$

Thus $\frac{d}{dp^2}(|v_n^1|^2\;\tilde{\mathrm{E}}_n^1)$ vanishes to linear order in the energy correction, as expected from the arguments of section 2.2. The corrections to the matrix elements is $O(1/(\sigma T^2)^2)$.

3 dimensions

In 3 dimensions, the closed-string expansion reads [5]

$$\mathcal{Z}(r,T) = e^{-\mu T}\sqrt{\frac{T}{2r}}\sum_{n\geq 0} w_n e^{-\tilde{M}_n^0 r}\left\{1-[\tilde{M}_n^1]r - c_2\left[\frac{4\pi}{T^2}(n-\frac{1}{24})+\frac{1}{4Tr}\right]\right\}. \tag{4.11}$$

The calculation is slightly more involved than in 4d. One can first write the Fourier coefficient in the form:

$$c_1(p,r,T) \;=\; e^{-\mu T}\;\sqrt{\frac{Tr}{4\pi}}\;\sum_{n\geq 0} w_n \;\times \tag{4.12}$$
$$\left\{1-\frac{4\pi c_2}{T^2}(n-\frac{1}{24})+r[E_n^1]\frac{\partial}{\partial u}-\frac{c_2}{4Tr}\int_{r\tilde{M}_n^0}^{\infty}du\right\}_{u=r\tilde{M}_n^0}^{v=pr} I^{(3)}(u,v)$$

where

$$I^{(3)}(u,v) \equiv \frac{u^{3/2}}{\sqrt{u^2+v^2}}\;\left(K_{\frac{3}{4}}(z_+)K_{\frac{1}{4}}(z_-)+K_{\frac{3}{4}}(z_-)K_{\frac{1}{4}}(z_+)\right), \tag{4.13}$$

$z_\pm$ being the solutions of the quadratic equation

$$z^2 \;-\; z\sqrt{u^2+v^2} \;+\; \frac{u^2}{4} \;=\; 0. \tag{4.14}$$

The primitive of $I^{(3)}(u,v)$ is known: $\frac{\partial}{\partial u}J^{(3)}(u,v) = I^{(3)}(u,v)$ with

$$J^{(3)}(u,v) = -2\sqrt{u}\;K_{\frac{1}{4}}(z_+)K_{\frac{1}{4}}(z_-). \tag{4.15}$$

The result can thus be expressed concisely as

$$c_1(p,r,T) = e^{-\mu T}\,\sqrt{\frac{Tr}{4\pi}}\,\sum_{n\geq 0} w_n\ \times \tag{4.16}$$
$$\left\{\left[1-\frac{4\pi c_2}{T^2}(n-\frac{1}{24})\right]\frac{\partial}{\partial u}+r[E_n^1]\frac{\partial^2}{\partial u^2}+\frac{c_2}{4Tr}\right\}_{u=r\tilde{M}_n^0}^{v=pr} J^{(3)}(u,v)$$

It is now most efficient to first expand $J^{(3)}(u,v)$ before applying the various derivatives. Requiring that the leading $1/r$ term (which comes from the first and the last term in the curly brackets) vanishes leads to the condition $c_2 = \frac{1}{2\sigma}$, thus confirming the result obtained by Lüscher and Weisz [5]. Some further algebra then leads again to the result (4.7) for the dispersion relation with the energy corrections given by

$$[\tilde{M}_n^1] = -\frac{(4\pi)^2}{2\sigma T^3}(n-\frac{1}{24})^2, \tag{4.17}$$

while the correction to the matrix elements is

$$|v_n^1(p,T)|^2 = |v_n^0|^2\,\left(1-\frac{2\pi}{\sigma T^2}(n-\frac{1}{24})+\frac{[\tilde{M}_n^1]}{8\tilde{M}_n^0}\frac{3p^2-5(\tilde{M}_n^0)^2}{p^2+(\tilde{M}_n^0)^2}\right) \tag{4.18}$$
$$= |v_n^0|^2\,\left(\left(1-\frac{\pi}{\sigma T^2}(n-\frac{1}{24})\right)^2+\left(\frac{2\pi}{\sigma T^2}(n-\frac{1}{24})\right)^2\frac{(\sigma T)^2-p^2}{(\sigma T)^2+p^2}\ +\ O(1/(\sigma T^2)^3)\right).$$

It is straightforward to verify that $\frac{d}{dp^2}(|v_n^1|^2\ \tilde{\mathrm{E}}_n^1)$ is again $O([\tilde{M}_n^1]^2)$.

In the light of the examples seen, it is clear that the general prescription to obtain the dispersion relation from the closed-string representation of the partition function can be given: a positive power m of r inside the curly braces of eq. (4.11) is replaced by $(-r\frac{\partial}{\partial u})^m$ inside the curly braces of eq. (4.12) and a negative power m of r is replaced by $\frac{r^{-m}}{m!}\int_{r\tilde{M}_n^0}^{\infty} du_1\ldots du_m$. In general, both lead to energy corrections and to terms containing a negative power of r. The latter must be required to cancel order by order.

5. The free string with one finite transverse dimension

In this section, we consider the possibility of having one dimension (say $\hat{1}$) transverse to the plane defined by the two Polyakov loops finite and of length L. The parameter L^{-1} may then also be interpreted as a temperature at which the gauge theory is probed. In that respect, we remark that studying the dynamics of the 'spatial' QCD string has the advantage (over the ordinary electric one) that the closed strings can be kept arbitrary long independently of L, while winding modes around that periodic dimension play an important role, as we shall see.

In [15], it was argued, using the XY model, that when L is finite but large compared to $\sigma^{-1/2}$, the massless fluctuations of the string are unaffected by the size of L: the corrections would appear only in terms in E_n suppressed by a power of R greater than 3. The string tension itself, however, becomes a function of L. Thus it is legitimate in this regime to investigate the properties of the 'spatial' Polyakov loop correlator using the free-string partition function.

It is then natural to ask what one obtains for $\mathcal{Z}_0$ if, working backwards, one uses the expression (3.2) for c_0 (which was derived from $\mathcal{Z}_0$ when $L = \infty$) and sums over the now discretized momenta $p_1 = 0, \pm\frac{2\pi}{L}, \ldots$ (compare with eq. (2.3)):

$$\mathcal{Z}_0(x_1, r, T, L) = e^{-\mu T} \pi T \sum_{n\geq 0} w_n \frac{1}{L} \sum_{p_1} e^{ip_1 x_1} \int_{-\infty}^{\infty} \frac{dp_2}{2\pi} \frac{e^{-r\sqrt{p_1^2+p_2^2+(\tilde{M}_n^0)^2}}}{\sqrt{p_1 + p_2^2 + (\tilde{M}_n^0)^2}}. \tag{5.1}$$

Upon integrating over p_2 and using Poisson's summation formula $\frac{1}{L}\sum_p = \sum_{m\in\mathbf{Z}} \int_{-\infty}^{\infty} \frac{dp}{2\pi}$, one then finds

$$\mathcal{Z}_0(x_1, r, T, L) = \sum_{m\in\mathbf{Z}} \mathcal{Z}_0(\sqrt{(mL + x_1)^2 + r^2}, T). \tag{5.2}$$

To interpret this result, let us choose $x_1 = 0$ and go to the open string representation of $\mathcal{Z}_0(r, T)$. The energy eigenstates are

$$E_{mn}(r, L) = \mu + \sigma(L)\sqrt{r^2 + m^2L^2} + \frac{\pi}{\sqrt{r^2 + m^2L^2}} \left(-\frac{1}{24}(d-2) + n\right), \quad n \geq 0, m \in \mathbf{Z} \tag{5.3}$$

and they have weight w_n for all m. In particular, the partition function (eq. (5.2)) is consistent with open-closed string duality. Given that the direction L was taken as periodic, it is clear that we have obtained a sum over classical configurations (labelled by m) of the open string joining the two static charges along straight paths winding an arbitrary number of times m around the periodic dimension, with vibrational states (labelled by n) built up on each of these classical string configurations. The same result (5.2) is obtained in three dimensions (starting from the asymptotic form (3.8)). The dual nature of the open and closed string expansions of $\mathcal{Z}_0$ is manifest here, for having *fewer* momenta in the closed string channel corresponds to *more* states in the open string channel.

We have however not taken into account the fact that the pure SU(N) gauge theory has a centre symmetry which is unbroken for $L \gg \sigma^{-1/2}$ and thus forbids states of different $\mathcal{N}$-ality to mix. Since the 'spatial' Polyakov loops are invariant under this symmetry, the open-string states can only have winding numbers that are multiples of N. On the closed-string side, this would however imply that the quantum of momentum has to be $\frac{2\pi}{NL}$ rather than $\frac{2\pi}{L}$. It is presently unclear to the author if the states with a fraction of the normal unit of momentum can be given a sensible interpretation, for instance as winding modes of the closed strings[2].

Finally, we remark that the winding modes do not generate a phase transition on the worldsheet. The reason is that their entropy factor is too weak. Rather we expect worldsheet-tearing configurations such as the vortices of the XY model to drive the phase transition [15] where the central charge of the string theory drops from $\frac{\pi}{12}$ to $\frac{\pi}{24}$ in $d = 4$ (or from $\frac{\pi}{24}$ to zero in $d = 3$). Provided the dimensionally reduced action for hot QCD correctly describes the asymptotic high-temperature behaviour of its magnetic observables, then this phenomenon must occur at a certain L^* [15]. This is currently being tested numerically [18].

[2] When $L < 1/T_c$, where T_c is the deconfining temperature of the gauge theory, the centre symmetry is broken, so this difficulty is absent.

6. Conclusion

We have shown explicitly that the effective string theory of Lüscher and Weisz predicts a relativistic dispersion relation for the closed winding flux states of SU(N) gauge theories in 3 and 4 dimensions up to two-loop order included. The matrix elements of Polyakov loop operators between them and the vacuum were derived as a by-product. Assuming the closed-string spectrum to be unchanged in the presence of a compact transverse dimension directly led to the existence of winding modes of the open strings around the compact dimension.

At a more general level, we have considered a long-distance effective theory for the Polyakov loop correlator in SU(N) gauge theory that imposes consistency with the spectral representation in the crossed channel (closed-string representation) order by order in the inverse Euclidean separation. We have shown that this is sufficient to insure relativistic kinematics for the energy eigenstates of that channel. It is well-known that bosonic string theory, as a theory valid for all distance scales, is only consistent with Lorentz symmetry in 26 dimensions. Nevertheless, in a non-critical number of dimensions, bosonic string theory can be reconciled with space-time symmetry order by order in an expansion around classical, elongated string configurations, a conclusion already reached in [16].

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References

[1] J. Kuti, *Lattice QCD and string theory*, *PoS* **LAT2005** (2006) 001 [hep-lat/0511023].

[2] M. Lüscher and P. Weisz, *Quark confinement and the bosonic string*, *JHEP* **07** (2002) 049 [hep-lat/0207003].

[3] P. Majumdar, *The string spectrum from large Wilson loops*, *Nucl. Phys.* **B 664** (2003) 213 [hep-lat/0211038].

[4] M. Caselle, R. Fiore, F. Gliozzi, M. Hasenbusch and P. Provero, *String effects in the Wilson loop: a high precision numerical test*, *Nucl. Phys.* **B 486** (1997) 245 [hep-lat/9609041].

[5] M. Lüscher and P. Weisz, *String excitation energies in* SU(N) *gauge theories beyond the free-string approximation*, *JHEP* **07** (2004) 014 [hep-th/0406205].

[6] M. Caselle, M. Panero and P. Provero, *String effects in Polyakov loop correlators*, *JHEP* **06** (2002) 061 [hep-lat/0205008].

[7] M. Caselle, M. Pepe and A. Rago, *Static quark potential and effective string corrections in the (2+1)D* SU(2) *Yang-Mills theory*, *JHEP* **10** (2004) 005 [hep-lat/0406008].

[8] M. Lüscher, K. Symanzik and P. Weisz, *Anomalies of the free loop wave equation in the WKB approximation*, *Nucl. Phys.* **B 173** (1980) 365;
M. Lüscher, *Symmetry breaking aspects of the roughening transition in gauge theories*, *Nucl. Phys.* **B 180** (1981) 317.

[9] K.J. Juge, J. Kuti and C. Morningstar, *Fine structure of the QCD string spectrum*, *Phys. Rev. Lett.* **90** (2003) 161601 [hep-lat/0207004].

[10] K.J. Juge, J. Kuti and C. Morningstar, *QCD string formation and the casimir energy*, hep-lat/0401032; *The casimir energy paradox of the QCD string*, *Nucl. Phys.* **129** *(Proc. Suppl.)* (2004) 686 [hep-lat/0310039].

[11] M. Caselle, M. Hasenbusch and M. Panero, *Comparing the Nambu-Goto string with lgt results*, *JHEP* **03** (2005) 026 [hep-lat/0501027].

[12] H. Meyer and M. Teper, *Confinement and the effective string theory in* $\mathrm{SU}(N \to \infty)$*: a lattice study*, *JHEP* **12** (2004) 031 [hep-lat/0411039].

[13] K.J. Juge, J. Kuti, F. Maresca, C. Morningstar and M.J. Peardon, *Excitations of torelon*, *Nucl. Phys.* **129** *(Proc. Suppl.)* (2004) 703 [hep-lat/0309180].

[14] K. Dietz and T. Filk, *On the renormalization of string functionals*, *Phys. Rev.* **D 27** (1983) 2944.

[15] H.B. Meyer, *Vortices on the worldsheet of the QCD string*, *Nucl. Phys.* **B 724** (2005) 432 [hep-th/0506034]; *Kosterlitz-thouless transition on the worldsheet of the QCD string*, *PoS* **LAT2005** (2006) 278 [hep-lat/0509081].

[16] J. Polchinski and A. Strominger, *Effective string theory*, *Phys. Rev. Lett.* **67** (1991) 1681.

[17] M. Billo and M. Caselle, *Polyakov loop correlators from D0-brane interactions in bosonic string theory*, *JHEP* **07** (2005) 038 [hep-th/0505201].

[18] P. Majumdar, H. B. Meyer, in preparation.